APPENDIX 3

$N = 4n + 2$ order Nyquist pair

1 - Initial factorization

Theorem 1 in Appendix 2 gives us, inter alia, the condition that a $4n + 2$ order $F(z)$ filter must meet in order to be at null ISI. It may be recalled that it is written as follows:

$$2F_0(z)\hat{F}_0(z) + F_1^2(z) + z^{-1}F_3^2(z) = \gamma z^{-n}\ , \tag{31}$$

where γ is a non-null constant and where the components $F_i(z)$ are all nth degree components apart from $F_3(z)$, and meet the relationships $\hat{F}_2(z) = F_0(z)$, $F_1(z) = \hat{F}_1(z)$ and $F_3(z) = \hat{F}_3(z)$.

Theorem 4 - *Let a_i, $i = 1, \ldots, n + 1$, $n \geq 0$, $n + 1$ be constant values. We consider the column vector $[F_0(z), F_1(z), F_2(z), F_3(z)]^T$ defined by the following equality:*

$$\begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \\ F_3(z) \end{bmatrix} = M(a_1)\ Z\ M(a_2)\ Z\ \ldots\ Z\ M(a_n)\ Z \begin{bmatrix} 1 \\ a_{n+1} \\ 1 \\ 0 \end{bmatrix}, \tag{32}$$

Then the filter $F(z)$ whose polyphase components are the values $F_i(z)$, $i = 1, \ldots, 4$, is a null ISI, $4n + 2$ order, monic, linear phase filter.

Reciprocally, any null ISI, linear phase, monic, $4n + 2$ order filter accepts a decomposition of this form.

Demonstration - For $n = 0$, the polyphase components are the constant polynomials $F_0(z) = 1$, $F_1(z) = a_1$, $F_2(z) = 1$ and $F_3(z) = 0$. These polynomials trivially verify the condition (31) as well as the conditions of symmetry and degree. We then have $F(z) = 1 + a_1z^{-1} + z^{-2}$. Let us now suppose that the components of a $4n + 2$ order $F(z)$ filter $F_i(z)$, $i = 0, \ldots, 3$ verify the condition (31) as well as the conditions of symmetry and degree. Let us build $L_i(z)$, $i = 0, \ldots, 3$ with:

$$\begin{bmatrix} L_0(z) \\ L_1(z) \\ L_2(z) \\ L_3(z) \end{bmatrix} = \mathrm{M}(a_1)\ Z \begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \\ F_3(z) \end{bmatrix}. \tag{33}$$

We have:

$$L_0(z) = F_0(z) - \frac{a_1^2}{2}z^{-1}F_2(z) + a_1z^{-1}F_3(z)\ , \tag{34}$$

$$L_1(z) = -a_1 F_0(z) - a_1 z^{-1} F_2(z) + (1 - \frac{a_1^2}{2}) z^{-1} F_3(z) \,, \qquad (35)$$

$$L_2(z) = -\frac{a_1^2}{2} F_0(z) + z^{-1} F_2(z) + a_1 z^{-1} F_3(z) \,, \qquad (36)$$

$$L_3(z) = (1 + \frac{a_1^2}{2}) F_1(z) \,. \qquad (37)$$

The verification of the conditions of degree of symmetry on the components $L_i(z)$, i = 0, ..., 3 is done almost immediately. By developing the analogous expression of (31) for the components $L_i(z)$, i = 0, ..., 3, we obtain:

$$2L_0(z)\hat{L}_0(z) + L_1^2(z) + z^{-1}L_3^2(z) = \frac{(2+a_1^2)^2}{4} z^{-1} \left(2F_0(z)\hat{F}_0(z) + F_1^2(z) + z^{-1}F_3^2(z)\right) \qquad (38)$$

According to (31), the direct part of the theorem is therefore demonstrated.

The reciprocal is demonstrated if it can be established that the relationships (34-37) can uniquely determine a coefficient a_1 and the components $F_i(z)$, i = 0, ..., 3 of a null ISI, linear phase, monic $F(z)$ filter. Since we should have $F_0(z = 0) = 1$ (since $F(z)$ is monic and symmetrical, its constant term is equal to 1), according to (35) we have $a_1 = -L_1(z = 0)$. If $a_1 = 0$, then we are in the particular case where $L_0(z)$ is an n degree term, the $n + 1$ degree term being zero, the symmetrical polynomial $L_1(z)$ has a null constant term and its highest degree term is null, and finally the constant term of $L_2(z)$ is null (but not its $n + 1$ degree term whose coefficient is 1). The relationships (34-37) are then inverted by $F_0(z) = L_0(z)$, $F_1(z) = L_3(z)$, $F_2(z) = L_2(z)$ quo z^{-1} and $F_3(z) = L_1(z)$ quo z^{-1}. It is now assumed that $a_1 \neq 0$ but this is not actually necessary. The formal inversion of the formulae (34-37) leads to:

$$F_0(z) = \frac{4}{(2 + a_1^2)^2} \left(L_0(z) - a_1 L_1(z) - \frac{a_1^2}{2} L_2(z)\right) \,, \qquad (39)$$

$$F_1(z) = \frac{2}{2 + a_1^2} L_3(z) \,, \qquad (40)$$

$$F_2(z) = \frac{4}{(2 + a_1^2)^2} \left(-\frac{a_1^2}{2} L_0(z) - a_1 L_1(z) + L_2(z)\right) \text{ quo } z^{-1} \,, \qquad (41)$$

$$F_3(z) = \frac{4}{(2 + a_1^2)^2} \left(a_1 L_0(z) + (1 - \frac{a_1^2}{2}) L_1(z) + a_1 L_2(z)\right) \text{ quo } z^{-1} \qquad (42)$$

It is then enough to show that the quotients by z^{-1} are exact. Since $F(z)$ is monic, $L_0(z = 0) =$

1, $L_1(z=0) = -a_1$ by defining of a_1 and according to the relationship (31), $L_2(z=0) = -a_1^2/2$. The straight line members of the equalities defining $F_2(z)$ and $F_3(z)$ therefore cancel out in $z = 0$ and the quotients are exact. It can then be ascertained that the polynomials F_i, $i = 0, \ldots, 3$ meet the appropriate conditions of symmetry.

2 - Equivalent decomposition

Noting the following matrix as $\mathbf{R}(\alpha)$:

$$\mathrm{R}(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (43)$$

and using the formula (32) we get:

$$\begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \\ F_3(z) \end{bmatrix} = \mathbf{P}^T\,\mathbf{R}(a_1)\,\mathbf{PZP}^T\,\mathbf{R}(a_2)\,\mathbf{PZP}^T \;\ldots\; \mathbf{PZP}^T\,\mathbf{R}(a_n)\,\mathbf{PZ} \begin{bmatrix} 1 \\ a_{n+1} \\ 1 \\ 0 \end{bmatrix}$$

In the sending diagram, *M*a is used to denote the block associated with the right side vector.

The matrix $\bar{\mathrm{Z}}$ is then introduced. This matrix is defined by:

$$\bar{\mathbf{Z}} = \mathbf{PZP}^T = \begin{bmatrix} \frac{1}{2}(1+z^{-1}) & 0 & \frac{1}{2}(-1+z^{-1}) & 0 \\ 0 & 0 & 0 & z^{-1} \\ \frac{1}{2}(-1+z^{-1}) & 0 & \frac{1}{2}(1+z^{-1}) & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad (44)$$

taking account of:

$$\mathrm{P}\begin{bmatrix} 1 \\ a_{n+1} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ a_{n+1} \\ 0 \\ 0 \end{bmatrix}, \qquad (45)$$

we obtain, with a multiplier factor g, the breakdown:

$$\begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \\ F_3(z) \end{bmatrix} = g\mathbf{P}^T\mathbf{R}(a_1)\,\bar{\mathbf{Z}}\,\mathbf{R}(a_2)\,\bar{\mathbf{Z}}\,\ldots\,\mathbf{R}(a_n)\,\bar{\mathbf{Z}} \begin{bmatrix} 1 \\ a_{n+1} \\ 0 \\ 0 \end{bmatrix}. \qquad (46)$$

Operational complexity

The equation (46) directly gives the structure of the making of the half-Nyquist sending and reception filters. It corresponds to a system with four inputs and four outputs which, in the diagrams of Figures 6 and 7, take the place of the set of four polyphase filters. It will be noted that, in the case of the sending filter, the outputs have to be inverted with respect to the writing (46) which on the contrary corresponds precisely to that of the reception filter. Apart

from this difference of detail, the structure for each of these $4n + 2$ order filters will therefore correspond to:

- n matrix blocks of the $\bar{\bar{Z}}$ and $\mathbf{R}(a_l)$ types,
- one $\mathbf{P}^T$ matrix block,
- one vector block $[1, a_{n+1}, 0, 0]^T$,
- a multiplier g which is to be taken into account only for sending or reception.

Let us now make a more detailed examination of the cost of making each of these elements.

- Matrices $\mathbf{R}(\alpha_l)$

The matrices $\mathbf{R}(\alpha_l)$, $1 \leq l \leq n$ are block matrices which can take the form:

$$\mathbf{R}(\alpha_l) = \begin{pmatrix} \mathbf{G}_l & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \tag{47}$$

where $\mathbf{I}$ is the unity matrix and $\mathbf{G}_l$ is the matrix of rotation for the angle α_l, namely:

$$\mathbf{G}_l = \begin{pmatrix} \cos(\alpha_l) & -\sin(\alpha_l) \\ \sin(\alpha_l) & \cos(\alpha_l) \end{pmatrix} \tag{48}$$

The circuit arrangement cost is therefore quite simply that of the lattice corresponding to the rotation $\mathbf{G}_l$, namely four multiplications and two additions.

- The matrix $\bar{Z}$.

It can be used for the association, with an input vector e, of the output vector s by the relationship $s = \bar{Z}e$ which is expressed by the following four equations:

$$\begin{aligned} s_0 &= \tfrac{1}{2}[(e_0 - e_2) + z^{-1}(e_0 + e_2)] \\ s_1 &= z^{-1}e_1 \\ s_2 &= \tfrac{1}{2}[(e_2 - e_0) + z^{-1}(e_0 + e_2)] \\ s_3 &= e_3 \end{aligned} \tag{49}$$

The structural diagram of a block of this kind is shown in Figure 14. It can be specified that the multiplications by 1/2 can be reduced through a simple shift of binary data. In addition to these two shifts, the cost is therefore simply four additions and one sign inversion.

- The matrix $\mathbf{P}^T$

It takes the form:

$$\mathbf{P}^T = \begin{pmatrix} a & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{50}$$

with $a = \frac{1}{\sqrt{2}}$. This embodiment can also be deduced from the input/output equations which in the present case are written as follows:

$$\begin{aligned} s_0 &= a(e_0 + e_2) \\ s_1 &= e_1 \\ s_2 &= a(-e_0 + e_2) \\ s_3 &= e_3 \end{aligned} \tag{51}$$

immediately it is therefore possible to then deduce the diagram of Figure 15.

The operational cost is two multiplications, two additions and one sign inversion.

- The vector block $[1, a_{n+1}, 0, 0]^T$

Its circuit arrangement cost is the equivalent of only one single multiplication.

The full circuit arrangement diagram according to the equation (46) therefore corresponds to a system of four inputs and four outputs as shown in Figure 16. Naturally, this system is found at sending and at reception. The order of complexity of each of these filters is therefore $4n$ MPU and $6n$ APU.

Thus, as already stated, at sending it is necessary to invert the outputs. In the drawing showing the making of the device in Figure 17, this operation is symbolized by the block **J** corresponding to the antidiagonal matrix, namely:

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{52}$$

The reception diagram for the $4n + 2$ order is shown in Figures 18 and 19. It is deduced from the following computations.

The matrix $\hat{\mathbf{Z}}$ is introduced, defined by :

$$\hat{\mathbf{Z}} = \begin{bmatrix} \frac{1}{2}(1 + z^{-1}) & 0 & \frac{1}{2}(1 - z^{-1}) & 0 \\ 0 & 0 & 0 & z^{-1} \\ \frac{1}{2}(1 - z^{-1}) & 0 & \frac{1}{2}(1 + z^{-1}) & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

It is the matrix obtained in taking the mirror elements of the elements of the matrix $\bar{\mathbf{Z}}$. We have:

$$\bar{\mathbf{Z}}\,\hat{\mathbf{Z}} = z^{-1}\mathbf{I}_4 ,$$

where $\mathbf{I}_4$ is the 4th order identity matrix.

Then the following identities are verified:

$$\mathrm{P}\ \mathrm{P}^T = \mathrm{I}_4 \ , \ \ \mathrm{R}(a)^{-1} = \mathrm{R}(a)^T = \mathrm{R}(-a)$$

for any value of a.

By noting the matrix product as **M**:

$$\mathrm{M} = \mathrm{P}^T\mathrm{R}(a_1)\bar{\mathrm{Z}}\ldots\mathrm{R}(a_n)\bar{\mathrm{Z}}$$

and the matrix product as **N** :

$$\mathrm{N} = \hat{\mathrm{Z}}\mathrm{R}(-a_n)\ldots\hat{\mathrm{Z}}\mathrm{R}(-a_1)\mathrm{P}$$

we therefore have:

$$\mathrm{NM} = z^{-n}\mathrm{I}_4 \ ,$$

and consequently :

$$\mathrm{N}\begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \\ F_3(z) \end{bmatrix} = gz^{-n}\begin{bmatrix} 1 \\ a_{n+1} \\ 0 \\ 0 \end{bmatrix} .$$

We then introduce the matrix **K** defined by:

$$\mathrm{K} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Figure 19 gives the drawing for the making of the block corresponding to the matrix **N** and Figure 20 is the for the making of the reception filter resulting therefrom. If the constant g is equal to $1/(1 + a_{n+1})$, then a signal transmitted by the sending system of Figure 17 and then received by the system of Figure 20 produces an identical signal with a delay of $n + 2$ samples.